A boundary integral method for computing eddy currents in non-manifold thin conductors

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We present a novel technique to solve eddy current problems in non-manifold thin conductors by a Boundary Integral Method (BIM) based on a stream function.

***Index Terms*—Boundary integral method (BIM), eddy currents, stream function, thin shields, non-manifold geometry**

I. Introduction

The Boundary Integral Method (BIM) represents a thin conductor as a 2d surface. Here, this approach is used to model a conductor that is thin with respect to the penetration depth of the magnetic field. In this case the induced current density can be considered as uniform in the conductor thickness and it can be modeled by a stream function [1]. Nonetheless, we remark that a stream function may be also used for problems with arbitrary skin depth [2]. Recently, an effective technique to render the stream function single valued based on the computation of cohomology generators in linear worst-case complexity has been introduced [3]. Nonetheless, the underlying hypothesis of [3] is to deal with a discrete surface, i.e. technically a combinatorial orientable 2-manifold with boundary embedded in $\mathbb{R}^3$ [4]. Even though some paper claim to extend BIM for arbitrary geometry and topology of the thin conductor, this is hardly plausible. Almost all papers assume the thin conductor to be an orientable 2-manifold, in most cases without saying it. Yet, non-manifold orientable surfaces arise frequently in practical engineering problems. To the best of our knowledge, this issue is discussed only in [1], [5], still without the necessary details. For example, how to automatically partition the geometry such that each partition is a topological disk is non-trivial and left completely unaddressed in [1], [5]. The aim of this contribution is to present an original and detailed recipe to generalize the use of the BIM to most non-manifold geometries.

II. BIM for manifold surfaces

The discrete surface is a mesh formed by N nodes $\{n_i\}_{i=1}^N$, E edges $\{e_j\}_{j=1}^E$ and F polygons $\{f_k\}_{k=1}^F$. Mesh incidences are encoded in the cell complex $\mathcal{K}$ [4]. Dual nodes $\{\tilde{n}_k\}_{k=1}^F$, dual edges $\{\tilde{e}_j\}_{j=1}^E$ and dual faces $\{\tilde{f}_i\}_{i=1}^N$ of the dual complex $\tilde{\mathcal{K}}$ are introduced [3]. Matrix $\mathbf{G}$ stores the edge-node incidences. We express the current per unit of thickness 1-cochain $\mathbf{I}$, with

$$\mathbf{I} = \mathbf{G}\boldsymbol{\Psi} + \mathbf{H}\mathbf{i}, \tag{1}$$

where $\boldsymbol{\Psi}$ is the 0-cochain whose coefficients are the values of the stream function sampled on mesh nodes, $\mathbf{i}$ is the array of *independent currents* and the columns of $\mathbf{H}$ store the representatives of $H^1(\mathcal{K} - \partial\mathcal{K})$ generators, see [3] for more details. Then, we enforce the discrete Faraday's law

$$\mathbf{G}^T\tilde{\mathbf{U}} + i\omega\tilde{\boldsymbol{\Phi}} = -i\omega\mathbf{G}^T\tilde{\mathbf{A}}_s, \tag{2}$$

where $\tilde{\mathbf{U}}$ is the electromotive force (e.m.f.) on dual edges, $\tilde{\boldsymbol{\Phi}}$ is the magnetic flux produced by eddy currents on dual faces and $\tilde{\mathbf{A}}_s$ is the circulation of the magnetic vector potential due to the source currents on dual edges. The two constitutive laws are expressed in the discrete setting as

$$\tilde{\mathbf{U}} = \mathbf{R}\mathbf{I} \quad \text{and} \quad \tilde{\mathbf{A}} = \mathbf{M}\mathbf{I}, \tag{3}$$

where $\mathbf{R}$ and $\mathbf{M}$ are the classical resistance mass matrix and the magnetic matrix [1], respectively. By substituting (1), (3) and $\tilde{\boldsymbol{\Phi}} = \mathbf{G}^T\tilde{\mathbf{A}}$ inside (2) and by defining $\mathbf{K} = \mathbf{R} + i\omega\mathbf{M}$,

$$\left(\mathbf{G}^T\mathbf{K}\mathbf{G}\right)\boldsymbol{\Psi} + \left(\mathbf{G}^T\mathbf{K}\mathbf{H}\right)\mathbf{i} = -i\omega\mathbf{G}^T\tilde{\mathbf{A}}_s. \tag{4}$$

E.m.f.s evaluated on $H_1(\tilde{\mathcal{K}}) \simeq H^1(\mathcal{K} - \partial\mathcal{K})$ generators are still undetermined and non-local Faraday's laws enforced on them have to be added

$$\mathbf{H}^T\mathbf{K}\mathbf{G}\boldsymbol{\Psi} + \mathbf{H}^T\mathbf{K}\mathbf{H}\mathbf{i} = -i\omega\mathbf{H}^T\tilde{\mathbf{A}}_s. \tag{5}$$

III. Potentials revisited

The stream function on non-manifold nodes has to be redefined, otherwise it leads to inconsistencies in discrete current continuity law. To solve this issue, we first separate the non-manifold surface $\mathcal{S}$ into a maximal set of manifold parts in the following way. By $\mathcal{E}$ let us denote the set of edges where more than two faces of the mesh meet. The original mesh is partitioned into pieces such that each piece is a manifold and the edges from $\mathcal{E}$ can appear only on boundary of each piece. For example, we consider the case where three manifold surfaces join together as in Fig. 1a. To construct a modified mesh we first triplicate the non-manifold edges and nodes and update the incidence matrices accordingly as in Fig. 1b.

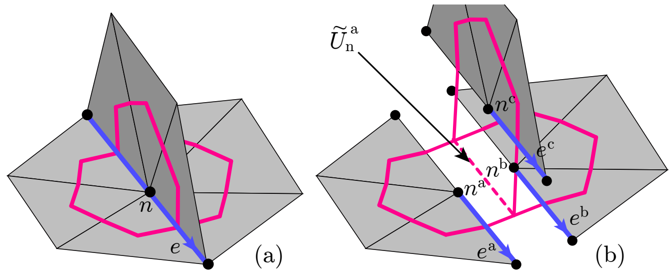

Fig. 1. (a) A non-manifold discrete surface $\mathcal{S}$. Non-manifold edges and nodes are marked. (b) $\mathcal{S}$ is split into maximal set of manifold parts. Each non-manifold node n is triplicated in nodes n^a, n^b and n^c. In the picture, the manifold parts are drawn exploded for clarity, but the coordinates of triplicated nodes are inherited from the original node. The additional dual edge, dotted in the picture, whose associated e.m.f. is $\tilde{U}_n^a$.

Second, we require the triplicated edges e^a, e^b and e^c to have the same orientation as the original edge e, see Fig. 1b. Finally, we also iso-orient all 2d elements on each manifold part. The detailed pseudo-code of the algorithm that performs these steps will be documented in the full paper.

Once triplicated edges are iso-oriented, the discrete continuity law $I_{e^a} + I_{e^b} + I_{e^c} = 0$ holds by enforcing the following constraint on the stream function on the nodes triplet $\{n^a, n^b, n^c\}$ corresponding to the non manifold node n

$$\Psi_{n^a} + \Psi_{n^b} + \Psi_{n^c} = 0. \tag{6}$$

All these constraints are collected together as $\mathbf{N\Psi} = \mathbf{0}$. If the orientation of the triplicated edges is not the same, it frequently leads to inconsistencies, i.e. two different constraints—similar to (6)—for the same node.

The dual complex of a manifold with boundary is open and it is customary to close it with an additional dual complex on the boundary of each manifold part, see the dotted edge in Fig. 1b. These additional dual edges are always present in the boundary of the dual complex, but usually they are not explicitly constructed because they are merely used to impose boundary conditions. That is, the dual complex $\tilde{\mathcal{K}}$ comprises also the additional dual edges on the boundary, which are dual to nodes. It will be shown that Ampère's law holds implicitly by constructing matrix $\mathbf{H}$ in (1) with the dual of generators of $H_1(\tilde{\mathcal{K}})$. We remark that for $H_1(\tilde{\mathcal{K}})$ computation the additional dual edges are not considered as they are not dual to any edge, and we need this duality to hold in (5).

IV. Novel formulation

We have to update Faraday's laws (2), as the dual cycles that correspond to non-manifold nodes comprise also the contribution of e.m.f.s on the additional dual edges on the boundary (as the dotted edge in Fig. 1b). Faraday's laws comprising the additional e.m.f.s become

$$\mathbf{G}^T\tilde{\mathbf{U}} + \mathbf{W}\tilde{\mathbf{U}}^a + i\omega\tilde{\mathbf{\Phi}} = -i\omega\mathbf{G}^T\tilde{\mathbf{A}}_s, \tag{7}$$

where matrix $\mathbf{W}$ stores the incidences between the additional dual edges and the dual cycles and $\tilde{\mathbf{U}}^a$ are the additional unknowns e.m.f.s (one for each non-manifold node n). To obtain a symmetric system, that is $\mathbf{W} = \mathbf{N}^T$, all incidences must be one. It will be shown that this implies a constraint on the normal of each manifold part. The detailed pseudo-code to select the right normal will be provided in the full paper. We remark that if there is some non-orientable part like a Möbius band embedded in the non-manifold surface, one cannot orient all manifold parts consistently and the software exits notifying the user about this problem.

The final system to solve is then

$$\begin{bmatrix} \mathbf{G}^T\mathbf{KG} & \mathbf{G}^T\mathbf{KH} & \mathbf{N}^T \\ \mathbf{H}^T\mathbf{KG} & \mathbf{H}^T\mathbf{KH} & \mathbf{0} \\ \mathbf{N} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{\Psi} \\ \mathbf{i} \\ \tilde{\mathbf{U}}^a \end{bmatrix} = \begin{bmatrix} -i\omega\mathbf{G}^T\tilde{\mathbf{A}}_s \\ -i\omega\mathbf{H}^T\tilde{\mathbf{A}}_s \\ \mathbf{0} \end{bmatrix}. \tag{8}$$

Concerning boundary conditions, the stream function on nodes on the boundary of $\mathcal{S}$ should be put to zero. Moreover, if some connected components of $\mathcal{S}$ are closed, the stream function on one arbitrary node of that connected component has to be fixed to zero.

V. Numerical results

A simple example is here considered to validate the theory: a circular loop (AC current, 50Hz) is placed around a non-manifold conducting surface, as shown in Fig. 2. More results will be presented in the full paper.

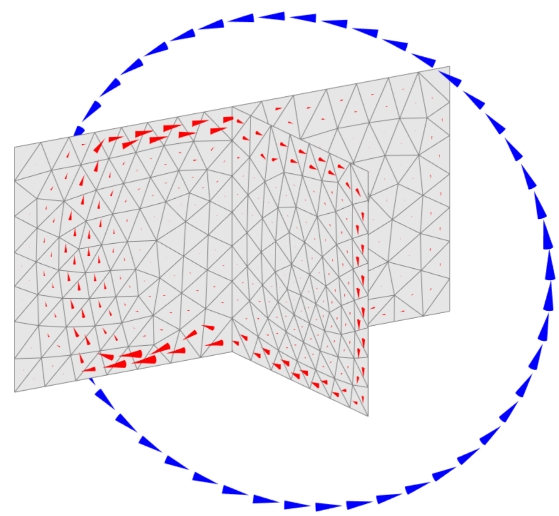

Fig. 2. A non-manifold discrete surface $\mathcal{S}$. Blue cones: source (circular loop, AC current at 50Hz). Red cones: real part of the current density (a.u.).

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